LOGIC SYNTHESIS AND TWO-LEVEL LOGIC OPTIMIZATION

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Outline

- Overview of logic synthesis.
- Combinational-logic design:
  - Background.
  - Two-level forms.
- Exact minimization.
- Covering algorithms.
- Boolean relations.

Logic synthesis and optimization

- Determine microscopic structure of the circuit.
- Explore *(area-delay)*trade-off:
  - Combinational circuits:
    - I/O delay.
  - Sequential circuits:
    - *cycle-time.*
- Explore *(power-delay)*trade-off:
- Enhance circuit *testability.*

Circuit implementation issues

- Implementation styles:
  - Two-level (e.g. PLA macro cells).
  - Multi-level (e.g. cell-based, array-based).
- Operation:
  - Combinational.
  - Sequential:
    - Synchronous
    - Asynchronous.

Design flow in logic synthesis

- Circuit capture:
  - Tabular specifications of functions or *finite-state machines* (FSMs).
  - Schematic capture.
  - Hardware Description Languages (HDLs).
- Synthesis and optimization:
  - Map circuit representation to abstract model.
  - Transformations on abstract model.
  - Library binding.

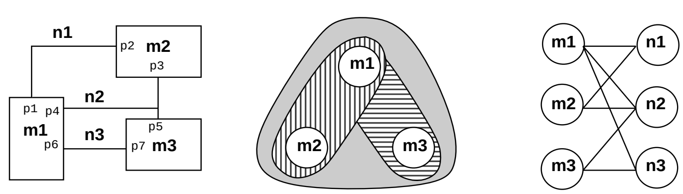
Abstract models

- Models based on graphs.
- Useful for:
  - Machine-level processing.
  - Reasoning about properties.
- Derived from language models by compilation.

Structural views

- Netlists:
  - Modules, nets, incidence.
  - Ports.
  - Hierarchy.
- Incidence (sparse) matrix of a graph.

Example



(a) (b) (c)

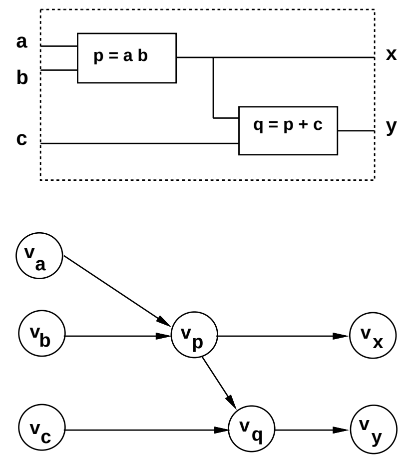
Logic funcions

- Black-box model of a combinational module.
- Defined on Boolean Algebra.
- *Support variables* correspond to module inputs.
- Logic functions may have multiple outputs and be *incompletely specified.*

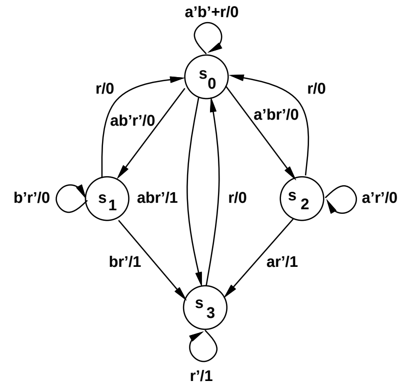
Logic networks

- Mixed structural/behavioral views.
- Useful for multiple-level logic (combinational and sequential).
- Interconnection of modules:
  - Logic gates.
  - Logic functions.

Example



State diagrams



- Model behavior of sequential circuits.
- Graph:
  - Vertices = states.
  - Edges = transitions.

Major logic synthesis problems

- Optimization of logic function representation.
  - Minimization of two-level forms.
  - Optimization of Binary Decision Diagrams (BDDs).
- Synthesis of combinational multiple-level logic networks.
  - Optimization or area, delay, power, testability.
- Optmization of FSM models.
  - State minimization, encoding.
- Synthesis of sequential multiple-level logic networks.
  - Optimization or area, delay, power, testability.
- Library binding.
  - Optimal selection of library cells.

Combinational logic design background

- Boolean algebra:
  - Quintuple $(B, +, \cdot, 0, 1)$
  - Binary Boolean algebra $B = \{0, 1\}$
- Boolean function:
  - Single output: $f : B^n \to B$.
  - Multiple output: $f : B^n \to B^m$.
  - Incompletely specified:
    - *don't care* symbol $*$.
    - $f : B^n \to \{0, 1, *\}^m$.

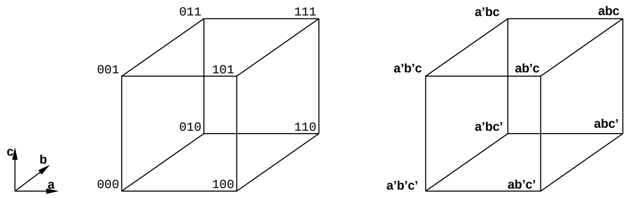
The *don't care* conditions

- We don't care about the value of the function.
- Related to the environment:
  - Input patterns that never occur.
  - Input patterns such that some output is never observed.
- Very important for synthesis and optimization.

Definitions

- Scalar function:
  - $ON-set$: subset of the domain such that f is true.
  - $OFF-set$: subset of the domain such that f is false.
  - $DC-set$: subset of the domain such that f is a *don't care* .
- Multiple-output function:
  - Defined for each component.

Cubical representation



Definitions

- Boolean *variables*.
- Boolean *literal*: variable and complement.
- *Product* or *cube*: product of literals.
- *Implicant*: product implying a value of a function (usually TRUE).
  - *Hypercube* in the Boolean space.
- *Minterm*: product of all input variables implying a value of a function (usually TRUE).
  - *Vertex* in the Boolean space.

Tabular representations

- *Truth table*:
  - List of all minterms of a function.
- *Implicant table* or *cover*:
  - List of implicants of a function sufficient to define function.
- Remark:
  - Implicant tables are smaller in size.

Example of truth table

$x = ab + a'c; \quad y = ab + bc + ac$

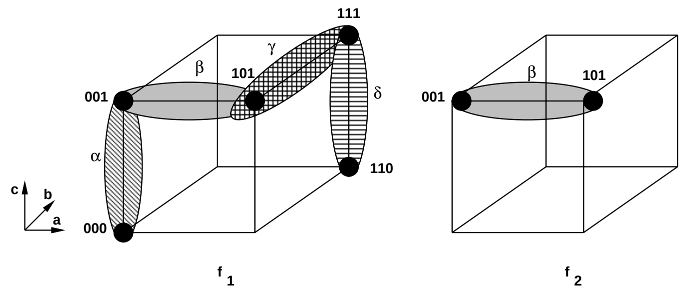
abc	xy
000	00
001	10
010	00
011	11
100	00
101	01
110	11
111	11

Example of implicant table

$x = ab + a'c; \quad y = ab + bc + ac$

abc	xy
001	10
*11	11
101	01
11*	11

Cubical representation of minterms and implicants



- $f_1 = a'b'c' + a'b'c + ab'c + abc + abc'$
- $f_2 = a'b'c + ab'c$

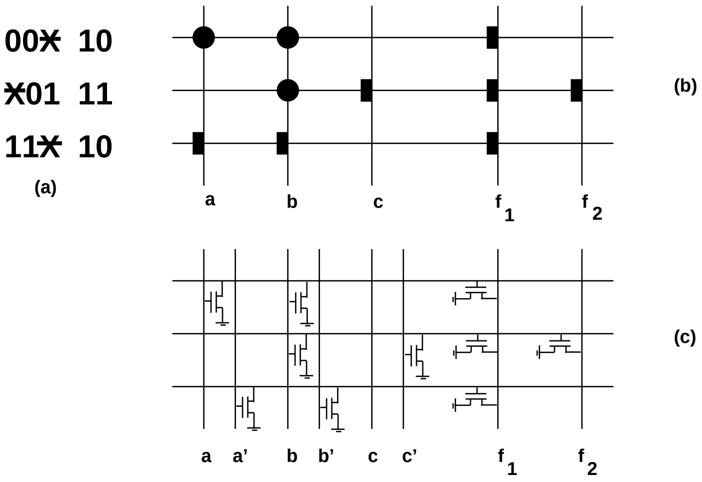
Two-level logic optimization motivation

- Reduce size of the representation.
- Direct implementation:
  - PLAs – reduce size and delay.
- Other implementation styles (e.g. multi-level):
  - Reduce amount of information.
  - Simplify local functions and connections.

Programmable logic arrays

- Macro-cells with rectangular structure.
- Implement any multi-output function.
- Layout easily generated by module generators.
- Fairly popular in the seventies/eighties (NMOS).
- Still used for control-unit implementation.

Programmable logic array



- $f_1 = a'b' + b'c + ab \qquad f_2 = b'c$

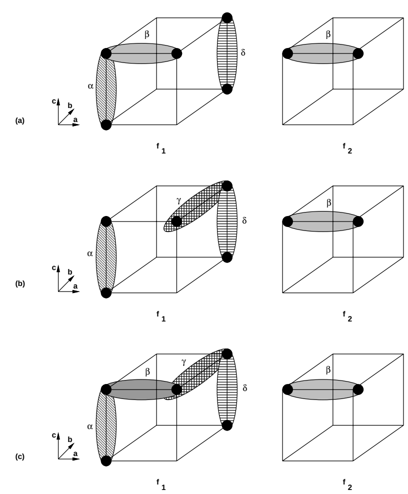
Two-level optimization

- Assumptions:
  - Primary goal is to reduce the number of implicants.
  - All implicants have the same cost.
  - Secondary goal is to reduce the number of literals.
- Rationale:
  - Implicants correspond to PLA rows.
  - Literals correspond to transistors.

Definitions

- *Minimum cover*:
  - Cover of the function with minimum number of implicants.
  - Global optimum.
- *Minimal cover* or *irredundant cover*:
  - Cover of the function that is not a proper superset of another cover.
  - No implicant can be dropped.
  - Local optimum.
- *Minimal cover w.r.t. 1-implicant containment.*
  - No implicant is contained by another one.
  - Weak local optimum.

Example



- $f_1 = a'b'c' + a'b'c + ab'c + abc + abc'$
- $f_2 = a'b'c + ab'c$

Definitions

- *Prime* implicant:
  - Implicant not contained by any other implicant.
- *Prime* cover:
  - Cover of prime implicants.
- *Essential* prime implicant:
  - There exist some minterm covered only by that prime implicant.

Logic minimization

- *Exact* methods:
  - Compute minimum cover.
  - Often impossible for large functions.
  - Based on *Quine McCluskey* method.
- *Heuristic* methods:
  - Compute minimal covers (possibly minimum).
  - Large variety of methods and programs:
    - MINI, PRESTO, ESPRESSO.

Exact logic minimization

- *Quine's theorem:*
  - There is a minimum cover that is prime.
- Consequence:
  - Search for minimum cover can be restricted to prime implicants.
- Quine McCluskey method:
  - Compute prime implicants.
  - Determine minimum cover.

Prime implicant table

- Rows: minterms.
- Columns: prime implicants.
- Exponential size:
  - 2^n minterms.
  - Up to $3^n/n$ prime implicants.
- Remark:
  - Some functions have much fewer primes.
  - Minterms can be grouped together.

Example

- Function: $f = a'b'c' + a'b'c + ab'c + abc + abc'$

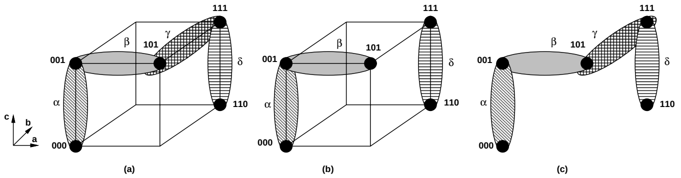
- Primes:

α	00*	1
β	*01	1
γ	1*1	1
δ	11*	1

- Implicant table:

	α	β	γ	δ
000	1	0	0	0
001	1	1	0	0
101	0	1	1	0
111	0	0	1	1
110	0	0	0	1

Example



Minimum cover early methods

- Reduce table:
  - Iteratively identify essentials, save them in the cover, remove covered minterms.

- *Petrick*'s method.
  - Write covering clauses in *pos* form.
  - Multiply out *pos* form into *sop* form.
  - Select cube of minimum size.
  - *Remark*:
    - Multiplying out clauses is exponential.

Example Petrick's method

- *pos* clauses:
  - $(\alpha)(\alpha+\beta)(\beta+\gamma)(\gamma+\delta)(\delta) = 1$

- *sop* form:
  - $\alpha\beta\delta + \alpha\gamma\delta = 1$

- Solutions:
  - $\{\alpha, \beta, \delta\}$
  - $\{\alpha, \gamma, \delta\}$

Matrix representation

- View table as Boolean matrix: **A**.
- Selection Boolean vector for primes: **x**.
- Determine **x** such that:
  - $\mathbf{A}\ \mathbf{x}\ \geq \mathbf{1}$.
  - Select enough columns to cover all rows.
- Minimize cardinality of **x**:
  - Example: $\mathbf{x} = [1101]^T$

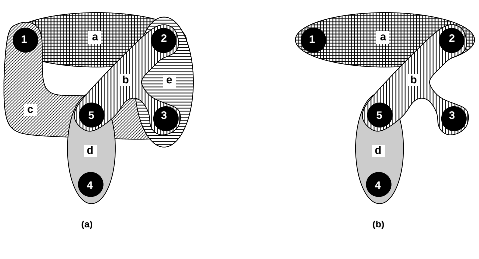
Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Covering problem

- Set covering problem:
  - A set S. (Minterm set).
  - A collection C of subsets. (Implicant set).
  - Select fewest elements of C to cover S.
- Intractable.
- Exact method:
  - Branch and bound algorithm.
- Heuristic methods.

Example edge-cover of a hypergraph



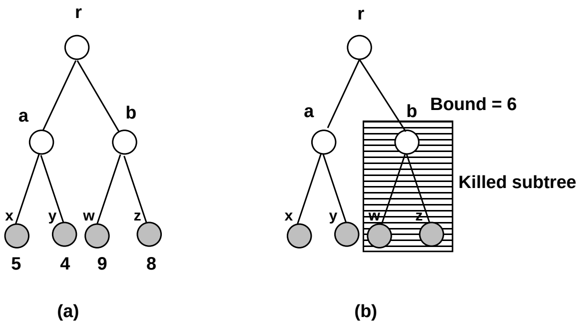
Branch and bound algorithm

- Tree search of the solution space:
  - Potentially exponential search.
- Use bounding function:
  - If the lower bound on the solution cost that can be derived from a set of future choices exceeds the cost of the best solution seen so far:
  - Kill the search.
- Good pruning may reduce run-time.

Branch and bound algorithm


```
BRANCH_AND_BOUND {
        Current_best = anything;
        Current_cost = ∞;
        S = s_0;
        while (S ≠ ∅) do {
                Select an element in s ∈ S;
                Remove s from S ;
                Make a branching decision based on s
                    yielding sequences {s_i, i = 1, 2, ..., m};
                for ( i = 1 to m) {
                        Compute the lower bound b_i of s_i;
                        if  (b_i ≥ Current_cost)
                                Kill s_i;
                        else {
                                if  (s_i is a complete solution ) {
                                        Current_best = s_i;
                                        Current_cost = cost of s_i ;
                                }
                                else
                                        Add s_i to set S;
                        }
                }
        }
}
```

Example



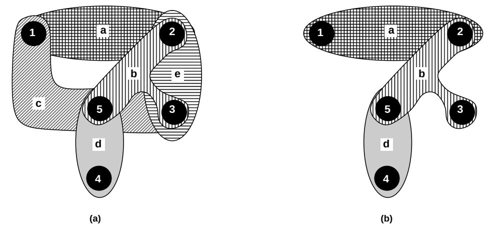
Branch and bound algorithm for covering Reduction strategies

- Partitioning:
  - If **A** is block diagonal:
    - ∗ Solve covering problem for corresponding blocks.
- Essentials (EPI):
  - Column incident to one (or more) row with single 1:
    - ∗ Select column.
    - ∗ Remove covered row(s) from table.

Branch and bound algorithm for covering Reduction strategies

- Column (implicant) dominance:
  - If $a_{ki} \geq a_{kj}\ \forall k$:
    - remove column j.
- Row (minterm) dominance:
  - If $a_{ik} \geq a_{jk}\ \forall k$:
    - Remove row i.

Example



$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example reduction

- Fourth column is essential.
- Fifth column is dominated.
- Fifth row is dominant.
- $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Branch and bound covering algorithm

$EXACT\_COVER(\mathbf{A}, \mathbf{x}, \mathbf{b})$ {
  Reduce matrix **A** and update corresponding **x**;
  if $(Current\_estimate \geq |\mathbf{b}|)$ **return**($\mathbf{b}$);
  if (**A** has no rows) **return** ($\mathbf{x}$);
  Select a branching column c;
  $x_c = 1$;
  $\widetilde{\mathbf{A}} = \mathbf{A}$ after deleting c and rows incident to it;
  $\widetilde{\mathbf{x}} = EXACT\_COVER(\widetilde{\mathbf{A}}, \mathbf{x}, \mathbf{b})$;
  if ($|\widetilde{\mathbf{x}}| < |\mathbf{b}|$)
    $\mathbf{b} = \widetilde{\mathbf{x}}$;
  $x_c = 0$;
  $\widetilde{\mathbf{A}} = \mathbf{A}$ after deleting c ;
  $\widetilde{\mathbf{x}} = EXACT\_COVER(\widetilde{\mathbf{A}}, \mathbf{x}, \mathbf{b})$;
  if ($|\widetilde{\mathbf{x}}| < |\mathbf{b}|$)
    $\mathbf{b} = \widetilde{\mathbf{x}}$;
  return ($\mathbf{b}$);
}

Bounding function

- Estimate lower bound on the covers derived from the current $\mathbf{x}$.

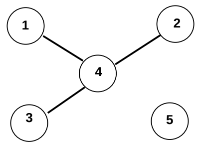
- The sum of the ones in $\mathbf{x}$, plus bound on cover for local $\mathbf{A}$:
  - Independent set of rows:
    - No 1 in same column.
  - Build graph denoting pairwise independence.
  - Find clique number.
  - Approximation (lower) is acceptable.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

- Row 4 independent from 1,2,3.

- Clique number is 2.



- Bound is 2.

Example

- $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

- There are no independent rows.

- Clique number is 1 (one vertex).

- Bound is $1 + 1$ (already selected essential).

Example

- $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

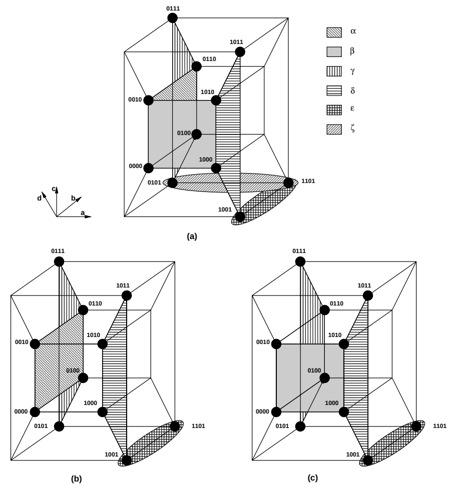
- Choose first column:
  - Recur with $\widetilde{\mathbf{A}} = [11]$.
    - Delete one dominated column.
    - Take other column (essential).
  - New cost is 3.

- Exclude first column:
  - Find another solution with cost 3 (discarded).

ESPRESSO-EXACT

- Exact minimizer [Rudell].
- Exact *branch and bound* covering.
- Compact implicant table:
  - Group together minterms covered by the same implicants.
- Very efficient. Solves most problems.

Example



α	0**0	1
β	*0*0	1
γ	01**	1
δ	10**	1
ϵ	1*01	1
ζ	*101	1

Example
Prime implicant table
(after removing essentials)

	α	β	ϵ	ζ
0000,0010	1	1	0	0
1101	0	0	1	1

Recent developments

- Many minimization problems can be solved exactly today.
- Usually bottleneck is table size.
- Implicit representation of prime implicants:
  - Methods based on BDDs [COUDERT]:
    - To represent sets.
    - To do dominance simplification.
  - Methods based on signature cubes [MCGEER]
    - Represent set of primes.

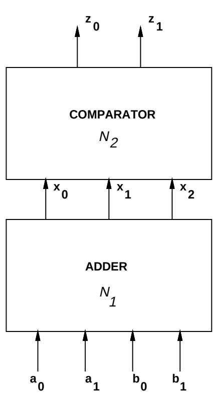
Summary
Exact two-level minimization of logic functions

- Based on derivatives of Quine-McCluskey method.
- Many minimization problems can be now solved exactly.
- Usual problems are memory size and time.

Boolean relations

- Generalization of Boolean functions.
- More than one output pattern may correspond to an input pattern.
- Some *degrees of freedom* in finding an implementation:
  - More general than *don't care* conditions.
- Problem:
  - Given a Boolean relation,
    find minimum cover of a compatible function.

Example



- Compare:
  - $\mathbf{a}+\mathbf{b} > 4$?
  - $\mathbf{a}+\mathbf{b} < 3$?

Example

a_1	a_0	b_1	b_0	**x**
0	0	0	0	{ 000, 001, 010 }
0	0	0	1	{ 000, 001, 010 }
0	0	1	0	{ 000, 001, 010 }
0	1	0	0	{ 000, 001, 010 }
1	0	0	0	{ 000, 001, 010 }
0	1	0	1	{ 000, 001, 010 }
0	0	1	1	{ 011, 100 }
0	1	1	0	{ 011, 100 }
1	0	0	1	{ 011, 100 }
1	0	1	0	{ 011, 100 }
1	1	0	0	{ 011, 100 }
0	1	1	1	{ 011, 100 }
1	1	0	1	{ 011, 100 }
1	0	1	1	{ 101, 110, 111 }
1	1	1	0	{ 101, 110, 111 }
1	1	1	1	{ 101, 110, 111 }

Example (2)
Minimum implementation

a_1	a_0	b_1	b_0	x
0	*	1	*	010
1	*	0	*	010
1	*	1	*	100
*	*	*	1	001
*	1	*	*	001

- Remark:
  - Circuit is no longer an adder.

Minimization of Boolean relations

- Since there are many possible output values there are many logic functions implementing the relation.
  - Compatible functions.
- Find a function with minimum cardinality.
- Do not enumerate all possible functions:
  - May be too many.
- Represent the primes of all possible functions:
  - Compatible primes ($c - primes$).

Minimization of Boolean relation

- Exact:
  - Find a set of compatible primes.
  - Solve a *binate* covering problem.
    * Consistency relations.
- Heuristic:
  - Iterative improvement [GYOCRO].

Example

- Boolean relation:

0	0	0	{ 00 }
0	0	1	{ 00 }
0	1	0	{ 00 }
0	1	1	{ 10 }
1	0	0	{ 00 }
1	0	1	{ 01 }
1	1	0	{ 00,11 }
1	1	1	{ 00,11 }

- Compatible primes:

α	0	1	1	10
β	1	0	1	01
γ	1	1	0	11
δ	1	1	1	11
ϵ	*	1	1	10
ζ	1	*	1	01
η	1	1	*	11

Example

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- Input 011 – output 10.
  - Covering clause $(\alpha + \epsilon)$.
- Input 111 – output 00 or 11.
  - No implicant – 00 – correct.
  - Either η or $\epsilon \cup \zeta$ – output 11 – correct.
  - Only ϵ or ζ is selected – output 10 or 01 – WRONG.
  - Covering clause $\eta + \epsilon\zeta + \epsilon'\zeta'$ – binate.
- Overall covering clause:

  $(\alpha + \epsilon) \cdot (\beta + \zeta) \cdot (\epsilon + \zeta' + \eta) \cdot (\epsilon' + \zeta + \eta)$

Binate covering

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- Covering problem with *binate clause.*
- Implications:
  - The selection of a prime may exclude other primes.
- No guarantee of finding a feasible solution:
  - Inconsistent clauses.
- *Minimum-cost satisfiability* problem.
  - Much harder to solve than unate cover.
  - Branch and bound algorithm.
  - BDD-based methods.

Summary Boolean relations

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- Generalization of Boolean functions.
  - Many possible output patterns.
- Useful for modeling:
  - Cascaded blocks.
  - Portions of multiple-level networks.
- More degree of freedom in implementation.
- Harder problem to solve.