FINITE-STATE MACHINE OPTIMIZATION

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Outline

- Modeling synchronous circuits:
  - *State-based* models.
  - *Structural* models.
- State-based optimization methods:
  - State minimization.
  - State encoding.

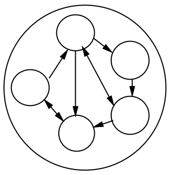
Synchronous Logic Circuits

- Interconnection of:
  - Combinational logic gates.
  - Synchronous delay elements:
    - E-T or M-S registers.
- Assumptions:
  - No direct combinational feedback.
  - Single-phase clocking.

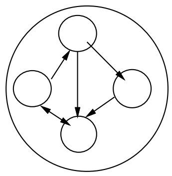
Modeling synchronous circuits

- *State-based* model:
  - Model circuits as *finite-state machines*.
  - Represent by *state tables/diagrams*.
  - Apply exact/heuristic algorithms for:
    - *State minimization*.
    - *State encoding*.
- *Structural* models:
  - Represent circuit by synchronous logic network.
  - Apply:
    - *Retiming*.
    - *Logic transformations*.

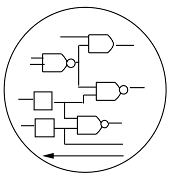
State-based optimization



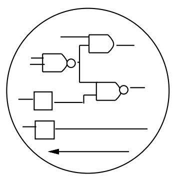
FSM Specification



State Minimization



State Encoding



Combinational Optimization

Formal *finite-state machine* model

- A set of primary inputs patterns X.
- A set of primary outputs patterns Y.
- A set of states S.
- A *state transition* function:
  - $\delta : X \times S \to S$.
- An *output function*:
  - $\lambda : X \times S \to Y$ for *Mealy* models
  - $\lambda : S \to Y$ for *Moore* models.

State minimization

- Completely specified *finite-state machines* :
  - No *don't care* conditions.
  - Easy to solve.
- Incompletely specified *finite-state machines* :
  - Unspecified transitions and/or outputs.
  - Intractable problem.

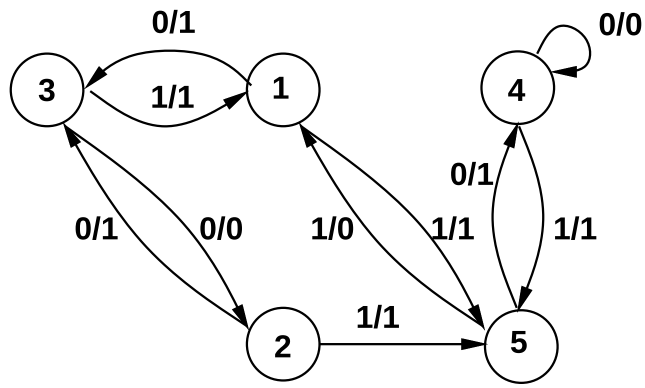
State minimization for completely specified FSMs

- Equivalent states:
  - Given any input sequence the corresponding output sequences match.
- Theorem:
  - Two states are equivalent iff:
    - they lead to identical outputs and their next-states are equivalent.
- Equivalence is transitive:
  - Partition states into *equivalence classes*.
  - *Minimum finite-state machine* is *unique*.

Example

INPUT	STATE	N-STATE	OUTPUT
0	s_1	s_3	1
1	s_1	s_5	1
0	s_2	s_3	1
1	s_2	s_5	1
0	s_3	s_2	0
1	s_3	s_1	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_1	0

Example



Algorithm

- Stepwise partition refinement.
- Initially:
  - All states in the same partition block.
- Then:
  - Refine partition blocks.
- At convergence:
  - Blocks identify equivalent states.

Algorithm

- $\Pi_1 =$ States belong to the same block when outputs are the same for any input.
- While further splitting is possible:
  - $\Pi_{k+1} =$ States belong to the same block if they were previously in the same block and their next-states are in the same block of Π_k for any input.

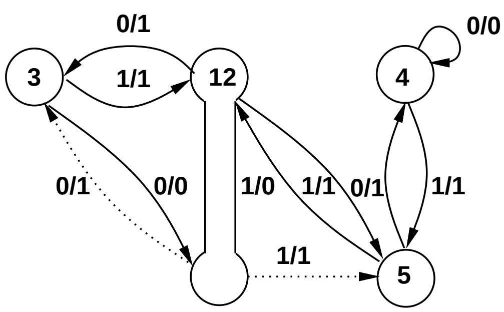
Example

- $\Pi_1 = \{\{s_1, s_2\}, \{s_3, s_4\}, \{s_5\}\}$.
- $\Pi_2 = \{\{s_1, s_2\}, \{s_3\}, \{s_4\}, \{s_5\}\}$.
- $\Pi_2 =$ is a partition into equivalence classes:
  - States $\{s_1, s_2\}$ are equivalent.

Example
minimal *finite-state machine*

INPUT	STATE	N-STATE	OUTPUT
0	s_{12}	s_3	1
1	s_{12}	s_5	1
0	s_3	s_{12}	0
1	s_3	s_{12}	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_{12}	0

Example



Computational complexity

- Polynomially-bound algorithm.
- There can be at most $|S|$ partition refinements.
- Each refinement requires considering each state:
  - Complexity $O(|S|^2)$.
- Actual time may depend upon:
  - Data-structures.
  - Implementation details.

State minimization for incompletely specified FSMs

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- *Applicable* input sequences:
  - All transitions are specified.
- *Compatible* states:
  - Given any applicable input sequence the corresponding output sequences match.
- Theorem:
  - Two states are compatible iff:
    - they lead to identical outputs
      - (when both are specified)
    - and their next-states are compatible
      - (when both are specified).

State minimization for incompletely specified FSMs

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- Compatibility is not an *equivalency* relation.
- *Minimum finite-state machine* is not *unique*.
- Implication relations make problem intractable.

Example

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INPUT	STATE	N-STATE	OUTPUT
0	s_1	s_3	1
1	s_1	s_5	*
0	s_2	s_3	*
1	s_2	s_5	1
0	s_3	s_2	0
1	s_3	s_1	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_1	0

Trivial method for the sake of illustration

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- Consider all the possible *don't care* assignments
  - n *don't care* imply
    - 2^n completely specified FSMs.
    - 2^n solutions.
- Example:
  - Replace * by 1.
    - $\Pi = \{\{s_1, s_2\}, \{s_3\}, \{s_4\}, \{s_5\}\}$.
  - Replace * by 0.
    - $\Pi = \{\{s_1, s_5\}, \{s_2, s_3, s_4\}\}$.

Compatibility and implications Example

- Compatible states $\{s_1, s_2\}$.
- If $\{s_3, s_4\}$ are compatible:
  - then $\{s_1, s_5\}$ are compatible.
- Incompatible states $\{s_2, s_5\}$.

Compatibility and implications

- Compatible pairs:
  - $\{s_1, s_2\}$
  - $\{s_1, s_5\} \Leftarrow \{s_3, s_4\}$
  - $\{s_2, s_4\} \Leftarrow \{s_3, s_4\}$
  - $\{s_2, s_3\} \Leftarrow \{s_1, s_5\}$
  - $\{s_3, s_4\} \Leftarrow \{s_2, s_4\} \cup \{s_1, s_5\}$
- Incompatible pairs:
  - $\{s_2, s_5\}, \quad \{s_3, s_5\}$
  - $\{s_1, s_4\}, \quad \{s_4, s_5\}$
  - $\{s_1, s_3\}$

Compatibility and implications

- A *class of compatible states* is such that all state pairs are compatible.
- A class is *maximal*:
  - If not subset of another class.
- Closure property:
  - A set of classes such that all compatibility implications are satisfied.
- The set of maximal compatibility classes:
  - Has the closure property.
  - May not provide a *minimum* solution.

Maximal compatible classes

- $\{s_1, s_2\}$
- $\{s_1, s_5\} \Leftarrow \{s_3, s_4\}$
- $\{s_2, s_3, s_4\} \Leftarrow \{s_1, s_5\}$
- Cover with MCC has cardinality 3.

Formulation of the state minimization problem

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- A class is prime, if not subset of another class implying the same set or a subset of classes.
- Compute the prime compatibility classes.
- Select a minimum number of PCC such that:
  - all states are covered.
  - all implications are satisfied.
- *Binate covering problem.*

Prime compatible classes

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- $\{s_1, s_2\}$
- $\{s_1, s_5\} \Leftarrow \{s_3, s_4\}$
- $\{s_2, s_3, s_4\} \Leftarrow \{s_1, s_5\}$
- Minimum cover: $\{\{s_1, s_5\}, \{s_2, s_3, s_4\}\}$.
- Minimum cover has cardinality 2.

Heuristic algorithms

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- Approximate the covering problem.
  - Preserve closure property.
  - Sacrifice minimality.
- Consider all maximal compatibility classes.
  - May not yield minimum.

State encoding

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- Determine a binary encoding of the states:
  - that optimize machine implementation:
    * area.
    * cycle-time.
- Modeling:
  - Two-level circuits.
  - Multiple-level circuits.

Two-level circuit models

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- *Sum of product* representation.
  - PLA implementation.
- Area:
  - # of products $\times$ # I/Os.
- Delay:
  - Twice # of products $plus$ # I/Os.
- Note:
  - # products of a *minimum* implementation.
  - # I/Os depends on encoding length.

State encoding for two-level models

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- Symbolic minimization of state table.
- Constrained encoding problems.
  - Exact and heuristic methods.
- Applicable to large *finite-state machines* .

Symbolic minimization

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- Extension of two-level logic optimization.
- Reduce the number of rows of a table, that can have symbolic fields.
- Reduction exploits:
  - Combination of input symbols in the same field.
  - Covering of output symbols.

State encoding of *finite-state machines*

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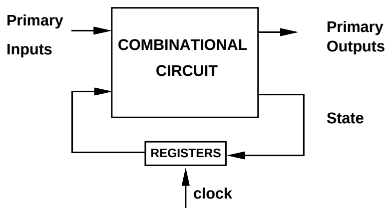
- Given a (minimum) state table of a *finite-state machine* :
  - find a consistent encoding of the states
    - that preserves the cover minimality
    - with minimum number of bits.

Example



INPUT	P-STATE	N-STATE	OUTPUT
0	s_1	s_3	0
1	s_1	s_3	0
0	s_2	s_3	0
1	s_2	s_1	1
0	s_3	s_5	0
1	s_3	s_4	1
0	s_4	s_2	1
1	s_4	s_3	0
0	s_5	s_2	1
1	s_5	s_5	0

Example

- Minimum symbolic cover:

$$\begin{array}{llll} * & s_1 s_2 s_4 & s_3 & 0 \\ 1 & s_2 & s_1 & 1 \\ 0 & s_4 s_5 & s_2 & 1 \\ 1 & s_3 & s_4 & 1 \end{array}$$

- Covering constraints:
  - s_1 and s_2 cover s_3
  - s_5 is covered by all other states.
- Encoding constraint matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

- Encoding matrix (one row per state):

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Encoded cover of combinational component:

$$\begin{array}{llll} * & 1{*}{*} & 001 & 0 \\ 1 & 101 & 111 & 1 \\ 0 & {*}00 & 101 & 1 \\ 1 & 001 & 100 & 1 \end{array}$$

Multiple-level circuit models

- *Logic network* representation.
  - Logic gate interconnection.
- Area:
  - # of literals.
- Delay:
  - Critical path length.
- Note
  - # literals and CP in a *minimum* network.

State encoding for multiple-level models

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- Cube-extraction heuristics [Mustang-Devadas].
- Rationale:
  - When two (or more) states have a transition to the same next-state:
    - Keep the distance of their encoding short.
    - Extract a large common cube.
- Exploit first stage of logic.
- Works fine because most FSM logic is shallow.

Example

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- 5-state FSM (3-bits).
  - $s_1 \to s_3$ with input i.
  - $s_2 \to s_3$ with input i'.
- Encoding:
  - $s_1 \to 000 = a'b'c'$.
  - $s_2 \to 001 = a'b'c$.
- Transition:
  - $ia'b'c' + i'a'b'c = a'b'(ic + i'c')$
  - 6 literals instead of 8.

Algorithm

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- Examine all state pairs:
  - Complete graph with $|V| = |S|$.
- Add weight on edges:
  - Model desired code proximity.
- Embed graph in the Boolean space.

Difficulties

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- The number of *occurrences* of common factors depends on the next-state encoding.
- The extraction of common cubes interact with each other.

Algorithm implementation

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- Fanout-oriented algorithm:
  - Consider present states and outputs.
  - Maximize the size of the most frequent common cubes.
- Fanin-oriented algorithm:
  - Consider next states and inputs.
  - Maximize the frequency of the largest common cubes.

Finite-state machine decomposition

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- Classic problem.
  - Based on partition theory.
  - Recently done at symbolic level.
- Different topologies:
  - Cascade, parallel, general.
- Recent heuristic algorithms:
  - Factorization [Devadas].

Example

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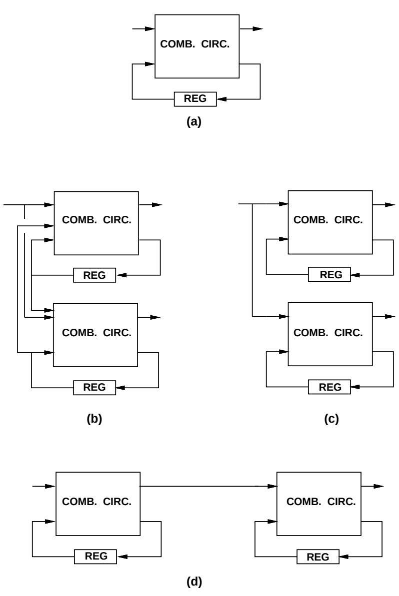

Summary

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- *Finite-state machine* optimization is commonly used.
  - Large body of research.
- State reduction/encoding correlates well to area minimization.
- Performance-oriented methods are still being researched.