Computer-Oriented Formulation of Transition-Rate Matrices via Kronecker Algebra

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Reader Aids—
Purpose: Widen state of the art
Special math needed for explanations: Matrix theory
Special math needed for results: Same
Results useful to: Reliability theoreticians

Abstract—This paper formulates the differential equations typical of a Markov problem in system-reliability theory in a systematic way in order to generate computer-oriented procedures. The coefficient matrix of these equations (the transition rate matrix) can be obtained for the whole system through algebraic operations on component transition-rate matrices. Such algebraic operations are performed according to the rules of Kronecker Algebra. We consider system reliability and availability with stress dependence and maintenance policies.

Theorems are given for constructing the system matrix in four cases:

- Reliability and availability with on-line multiple or single maintenance.
- Reliability and availability with system-state dependent failure rates.
- Reliability and availability with standby components.
- Off-line maintainability.

The results are expressed in algebraic terms and as a consequence their implementation by a computer program is straightforward. We also obtain information about the structure of the matrices involved. Such information can considerably improve computational efficiency of the computer codes because it allows introducing special ideas and techniques developed for large-system analysis such as sparsity, decomposition, and tearing.

1. INTRODUCTION

The purpose of this paper is to describe a computer-oriented approach to reliability analysis of complex systems. It is restricted to systems modeled as Markov processes. The Markov model leads to a set of linear homogeneous differential equations, such as $\dot{p}(t) = \Lambda p(t)$. The main result of this paper is a set of rules concerning the structure of the coefficient matrix of such a system. These rules are the starting point for a number of procedures to obtain the system matrix from the knowledge of each component failure and repair rate, system structure, maintenance policy, and stress dependence of failure rates.

The main feature of our procedures is their suitability for computer implementation; the overall matrix is built on an initial component matrix by an iterative procedure which, step by step, adds to the existing one the matrices pertinent to the other components of the system. This special result and the general computer oriented formulation of the problem are made possible by the use of Kronecker Algebra. Kronecker Algebra is an easy language for formal description of reliability problems. A few results from graph theory are used here in connection with the concept of relation graph for the purpose of decomposition. A computer program has not been written.

The method of building the transition rate matrix using Kronecker Algebra was proposed for the first time in [1] and was extended by [2]. We begin with the results in [1] and extend them to various cases involving *s*-dependence using Kronecker Algebra.

2. PRELIMINARY CONSIDERATIONS

2.1 *Notation* (Adapted from [3])

N	number of components in the system
(i)	superscript, component index, $i = 1, \ldots, N$
k	subscript, subsystem index, denotes the cardinality of the subsystem comprising the first k components (according to index order i)
j	subscript, system-state index, $j = 1, \ldots, 2^N$ and subsystem-state index $j = 1, \ldots, 2^k$
S	denotes a system or a subsystem
$x^{(i)}$	state of component i; 1 is the good state, 0 the bad state
x_j	state of the system (or subsystem)
$p^{(i)}(t)$	column vector, top element is $\Pr\{x^{(i)} = 1\}$, bottom element is the complement
$p(t)$	column vector, element is $p_j(t) \equiv \Pr\{x_j = 1\}$
c	structure vector, element is $c_j = 1$ if $x_j = 1$, otherwise $c_j = 0$
$\lambda^{(i)}, \mu^{(i)}$	transition rates of element i
Λ	transition-rate matrix, element is $\lambda_{u,v}$ $u, v = 1, \ldots, 2^N$
$A(t), R(t)$	availability and reliability at time t
R_{kl}	linear space of $k \times l$ real matrices

2.2 *Kronecker Algebra*

For a complete explanation see [4-6].

Direct (Kronecker) Product

Let $A \in R_{kl}$ and $B \in R_{mn}$. The Direct Product of A and B, is the partitioned matrix [4]

$$A \otimes B \equiv \left| \begin{array}{c|c|c} a_{11}B & \cdots & a_{1l}B \\ \hline \vdots & & \\ \hline a_{k1}B & & a_{kl}B \end{array} \right| \quad (1)$$

Kronecker Sum

Let $A \in R_{nn}$ and $B \in R_{mm}$. The Kronecker Sum of A and B is

$$A \oplus B \equiv (A \otimes I_m) + (I_n \otimes B) \quad (2)$$

where I_k is the unit matrix of order k.

Property 1

a. $(A + B) \otimes C = A \otimes C + B \otimes C$ (3a)

b. $A \otimes (B + C) = A \otimes B + A \otimes C$

Property 2

$$(A \otimes B) \times (C \otimes D) = (A \times C) \otimes (B \times D) \quad (4)$$

The dimensions of the matrices are such as to give meaning to the various terms.

Property 3

$$z \equiv x \otimes y \quad (5a)$$

$$\frac{dz}{dt} = x \otimes \frac{dy}{dt} + \frac{dx}{dt} \otimes y \quad (5b)$$

x and y are differentiable time-dependent vectors.

Property 4

A_1 and A_2 are decomposable (*σ-decomposition*), namely

$$A_1 = L_1 + U_1;\ A_2 = L_2 + U_2;\ L \equiv L_1 \otimes L_2;$$

$$U \equiv U_1 \otimes U_2 \quad (6)$$

where L_1, L_2 are lower triangular matrices and U_1, U_2 are upper triangular matrices.

$$A \equiv A_1 \oplus A_2 = L + U \quad (7)$$

are lower and upper triangular respectively.

The property obviously extends to the sum of more than two matrices. In other words σ-decomposition and Kronecker-sum commute.

3. STATE-INDEPENDENT PARAMETERS

We summarize our main results from [1] in the following two theorems and refer to [1] for details and proofs.

Theorem 1

$$p(t) = \prod_{\otimes\, i=1}^{N} p^{(i)}(t) \equiv p^{(1)}(t) \otimes p^{(2)}(t) \otimes ... p^{(N)}(t) \quad (8)$$

This is well-known in probability theory [7]; it is stated here in terms of Kronecker Algebra.

Theorem 2

$$\Lambda = \sum_{\oplus\, i=1}^{N} \Lambda^{(i)} \equiv \Lambda^{(1)} \oplus \Lambda^{(2)} \oplus ... \oplus \Lambda^{(N)} \quad (9)$$

or, in a recursive way,

$$\Lambda_k = \Lambda_{k-1} \oplus \Lambda^{(k)} \quad (10a)$$

$$\Lambda_1 = \Lambda^{(1)} \quad (10b)$$

Remark 1

Kronecker Sum provides a unique binary ordering of system states; index j for a state is related to component state by the relation:

$$j = 2^N - \sum_{i=1}^{N} x^{(i)} 2^{i-1} - 1 \quad (11)$$

Property 1

In general any entry λ_{uv} of transition-rate matrix depends both on the set of $\lambda^{(i)}$ and $\mu^{(i)}$. Nevertheless among all the possible ordering of system states, there exist some for which the following property holds:

λ_{uv} depends only on the set of $\lambda^{(i)}$ ($\mu^{(i)}$) for $u > v$ ($u < v$)

Theorem 3

Property 1 holds for the ordering of system states introduced by (11).

Remarks 2

Another ordering of system states [8] for which property 1 holds is that defined by an increasing number of failed components.

The following three problems can be treated.

i. Reliability without maintenance

$$\Lambda^{(i)} \equiv \begin{matrix} -\lambda^{(i)} & 0 \\ \lambda^{(i)} & 0 \end{matrix} \quad (12)$$

ii. Availability with multiple on-line maintenance

$$\Lambda^{(i)} \equiv \begin{matrix} -\lambda^{(i)} & \mu^{(i)} \\ \lambda^{(i)} & -\mu^{(i)} \end{matrix} \quad (13)$$

iii. Maintainability off-line

$$\Lambda^{(i)} \equiv \begin{matrix} 0 & \mu^{(i)} \\ 0 & -\mu^{(i)} \end{matrix} \quad (14)$$

Theorem 2 allows the transition-rate matrix to be build in a systematic way. We get for instance:

$$\Lambda^{(1)} \equiv \begin{matrix} -\lambda^{(1)} & \mu^{(1)} \\ \lambda^{(1)} & -\mu^{(1)} \end{matrix} \qquad \Lambda^{(2)} \equiv \begin{matrix} -\lambda^{(2)} & \mu^{(2)} \\ \lambda^{(2)} & -\mu^{(2)} \end{matrix}$$

$$\Lambda = \Lambda^{(1)} \oplus \Lambda^{(2)} = \begin{matrix} -\lambda^{(1)} - \lambda^{(2)} & \mu^{(2)} & \mu^{(1)} & 0 \\ \lambda^{(2)} & -\lambda^{(1)} - \mu^{(2)} & 0 & \mu^{(1)} \\ \lambda^{(1)} & 0 & -\mu^{(1)} - \lambda^{(2)} & \mu^{(2)} \\ 0 & \lambda^{(1)} & \lambda^{(2)} & -\mu^{(1)} - \mu^{(2)} \end{matrix}$$

Theorem 1 allows the probabilistic behaviour of the system to be computed without requiring the knowledge of Λ. It is only necessary to integrate N differential systems $p^{(i)}(t) = \Lambda^{(i)} p^{(i)}(t)$ and to perform $N - 1$ Kronecker Products.

4. STATE-DEPENDENT PARAMETERS

4. *No maintenance (systems without maintenance)*

The transition-rate matrix is lower triangular. Due to the s-dependence assumption, the failure rate of each component changes with the operative state. For each component we need to specify 2^{N-1} values for its failure rate. An equivalent way of supplying the same information is to specify the reference failure rate $\lambda^{*(i)}$ and a set of stress adjustment factors β_{ij} for all states j in which component i is good. Usually the reference is the state in which all components are good. Then $\lambda^{*(i)} = \lambda_0^{(i)}$, and—

$$\beta_{ij} \equiv \lambda_j^{(i)}/\lambda^{*(i)}. \quad (15)$$

The number of stress adjustment factors is $N(2^{N-1} - 1)$. They can be arranged in a $N \times (2^{N-1} - 1)$ rectangular matrix B (the s-dependence matrix). Its entries are 1 if there is s-independence. Therefore in practical applications many stress adjustment factors are equal to one.

The B matrix is usually of large dimension and for our purpose only the knowledge of a proper submatrix of B is necessary. Define a square $N \times N$ matrix $\Gamma \equiv \{\gamma_{ir}\}$ such that—

$$\gamma_{ir} \equiv \lambda^{(i)}\big|_{x(r)=0}/\lambda_0^{(i)} \qquad \text{for } i, r = 1, 2, \ldots, N \quad (16)$$

where $\lambda^{(i)}\big|_{x(r)=0}$ is the failure rate of component i in the system state in which only component r is failed. Γ is the reduced s-dependence matrix. In other words Γ is obtained from B by selecting the N columns related to the states with only one component failed. From (11), (15), (16) it follows that:

$$\gamma_{ir} = \beta_{i,2^{r-1}}. \quad (17)$$

The knowledge of Γ is enough for our purpose, if the following assumption is made.

Chain s-dependence assumption: The stress effect on component i due to the failure of a set of components is representable as the chain product of the stress adjustment factors of component i pertinent to the failed components

Let $H = \{h_1, h_2, \ldots, h_r\}$ be the subset of the failed components in state j. The stress adjustment factor is:

$$\beta_{ij} = \gamma_{ih_1} \cdot \gamma_{ih_2} \quad \cdot \gamma_{ih_r}. \quad (18)$$

This assumption drastically reduces the amount of input data from $N(2^{N-1} - 1)$ to $N(N - 1)$; that is the exponential growth is reduced to polynomial growth.

A matrix $A \equiv \{a_{ir}\}$ can now be defined:

$$a_{ir} \equiv \begin{cases} 0, & \text{for } \gamma_{ir} = 1 \\ 1, & \text{otherwise.} \end{cases} \quad (19)$$

We will refer to it as the *relation matrix*. It can be viewed as the adjacency matrix of a direct graph $G(V, A)$ without self-loops that we can consider as the relation graph of the problem. The correspondence between s-dependence among components and the relation graph is:

- A vertex n_i of G is associated with component i
- An oriented arc (n_i, n_r) is associated with s-dependence of component i failure rate on component r state.

Given a couple of vertices n_i and n_r three cases are possible:

1. No arc exists between them.
2. There is one arc from n_i to n_r and another from n_r to n_i.

3. There is only one arc from n_i to n_r or from n_r to n_i.

Case 1 corresponds to *s*-independence between components *i* and *r*; case 2 corresponds to bilateral *s*-dependence, and case 3) corresponds to unilateral *s*-dependence. Case 3) is typical of standby problems. The formulation of Λ from the knowledge of $\Lambda^{(i)}$ and from the stress adjustment factors is presented as Theorems 4 and 5.

Theorem 4 (Reliability without maintenance)

Let there be chain *s*-dependence among components, and let

$$G_{ir} \equiv \begin{matrix} 1 & 0 \\ 0 & \gamma_{ir} \end{matrix} \qquad (20)$$

be the reduced stress-adjustment matrix of component i w.r.t. component *r*. Then the transition-rate matrix of system is:

$$\Lambda = \sum_{i=1}^{N} \left[\left(\prod_{\otimes\, r=1}^{i-1} G_{ir} \right) \otimes \Lambda^{(i)} \otimes \left(\prod_{\otimes\, r=i+1}^{N} G_{ir} \right) \right]. \qquad (21)$$

Example 1

Find the Λ matrix for evaluating reliability without maintenance in the following problem. The system is composed of two parallel 10 kW electric generators supplying power to a 10 kW load through a line (see Fig. 1a).

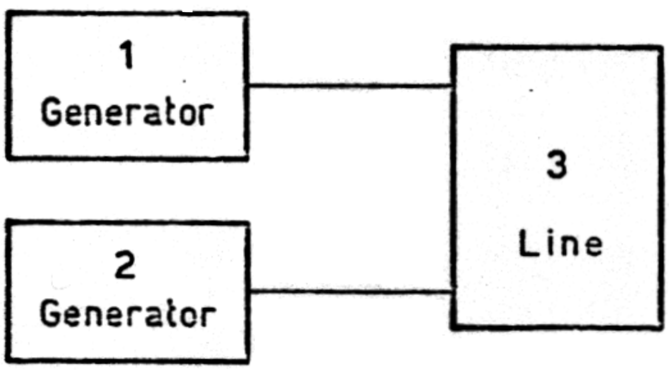

Fig. 1a. Power system block diagram.

A failure in one generator leaves the system working, though the other generator is working harder. A failure in the line (cutoff) causes a system failure, but lowers the generators failure rates. Fig. 1b shows the relation graph.

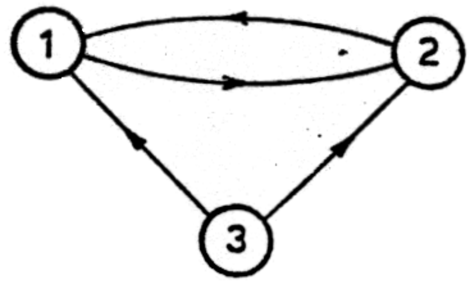

Fig. 1b. Relation graph.

Input data are:

$\lambda^{(1)} = \lambda^{(2)}$ generator failure rates

$\lambda^{(3)}$ electric line failure rate

$\gamma_{12} = \gamma_{21}$ stress adjustment factor representing the increased failure rate of a generator due to the failure of the other one

$\gamma_{13} = \gamma_{23}$ stress adjustment factor representing the decreased failure rate of both generators due to a failure (cutoff) of the line.

$$\Lambda^{(i)} = \begin{matrix} -\lambda^{(i)} & 0 \\ \lambda^{(i)} & 0 \end{matrix}, \quad \text{for } i = 1, 2, 3$$

$$G_{ir} = \begin{matrix} 0 & 0 \\ 0 & \gamma_{ir} \end{matrix}, \quad \text{for } (i, r) = (1, 2), (2, 1) \; (1, 3), (2, 3)$$

$$G_{32} = G_{31} = I_2$$

From (21):

$$\Lambda = \begin{bmatrix} -\lambda^{(1)}-\lambda^{(2)}-\lambda^{(3)} & & & & & & \bigcirc & \\ \lambda^{(3)} & -\gamma_{23}\lambda^{(2)}\gamma_{13}\lambda^{(1)} & & & & & & \\ \lambda^{(2)} & 0 & -\lambda^{(3)}-\gamma_{12}\lambda^{(1)} & & & & & \\ 0 & \gamma_{23}\lambda^{(2)} & \lambda^{(3)} & -\gamma_{13}\gamma_{12}\lambda^{(1)} & & & & \\ \lambda^{(1)} & 0 & 0 & 0 & -\gamma^{(3)}\gamma_{12}\lambda^{(1)} & & & \\ 0 & \gamma_{13}\lambda^{(1)} & 0 & 0 & \lambda^{(3)} & -\gamma_{21}\gamma_{23}\lambda^{(2)} & & \\ 0 & 0 & \gamma_{12}\lambda^{(1)} & 0 & \gamma_{21}\lambda^{(2)} & 0 & -\lambda^{(3)} & \\ 0 & 0 & 0 & \gamma_{13}\gamma_{12}\lambda^{(1)} & 0 & \gamma_{21}\gamma_{23}\lambda^{(2)} & \lambda^{(3)} & 0 \end{bmatrix}$$

Taking into account *s*-dependence implies managing a large amount of data. A first step to reduce the managing costs is through the chain-dependence assumption. A further reduction is possible when the following assumption can be made:

Unilateral s-dependence assumption

Let the relation graph of the problem be acyclic. Then it is possible to define an order of the nodes (and then of the components) such that no arc (n_i, n_r) exists if $i < r$. As a consequence, if $i < r$, then $\gamma_{ir} = 1$ and $G_{ir} = I_2$.

In this case, matrix Λ can be evaluated by means of Theorem 5.

Theorem 5 (Reliability without maintenance in unilateral *s*-dependence)

S_k is the subsystem containing the *k* first elements of the sequence 1, 2, ..., *N*. Λ_k is the transition rate matrix pertinent to S_k. Under chain *s*-dependence and unilateral *s*-dependence, Λ can be written as—

$$\Lambda_k = \Lambda_{k-1} \otimes I_2 + G_{kS_{k-1}} \otimes \Lambda^{(i)} \quad k = i = 2, 3, \ldots, N \qquad (22a)$$

$$\Lambda_1 = \Lambda^{(1)}$$

$$G_{kS_{k-1}} \equiv \prod_{\otimes\, r=2}^{i-1} G_{ir}$$

is the stress adjustment matrix representing the change in $\lambda^{(i)}$ due to the failure of subsystem S_{k-1}. If there is no state dependence, (22a) and (9) are equivalent. Because $G_{ir} = I_2$, for all $i, r = 1, 2, \ldots, N$ we have—

$$\Lambda_N = \Lambda_{N-1} \otimes I_2 + I_{2^{N-1}} \otimes \Lambda^{(N)} =$$

$$= \Lambda_{N-1} \oplus \Lambda^{(N)} = \sum_{\oplus\, i=1}^{N} \Lambda^{(i)}$$

Example 2

Find the Λ matrix for evaluating reliability without maintenance in the following problem.

A current-limited voltage regulator #2, feeds a microelectronic device #1, which contains about ten chips; the device might undergo a short circuit, but does not fail open-circuit. A standby i.i.d. regulator #3 is provided in case of failure of v.r. #2. $\lambda_0^{(1)} = 2 \times 10^{-6}$/hour and $\lambda_0^{(2)} = 0.1 \times 10^{-6}$/hour for the regulator at full load (Fig. 2a).

When regulator #3 is in standby (it is not working at full load because regulator #2 is working) its failure rate $\lambda_0^{(3)} = 0.05 \times 10^{-6}$/hour: since the failure at full load $\lambda^{(3)} = 0.1 \times 10^{-6}$/hour then $\gamma_{32} = 2$. The failure rate of device #1 does not depend on the state of the regulators. When device #1 is in short circuit, $\lambda^{(2)}$ and $\lambda^{(3)}$ are 10^{-6}/hour, i.e. $\gamma_{21} = \gamma_{31} = 10$. Fig. 2b shows the relation graph.

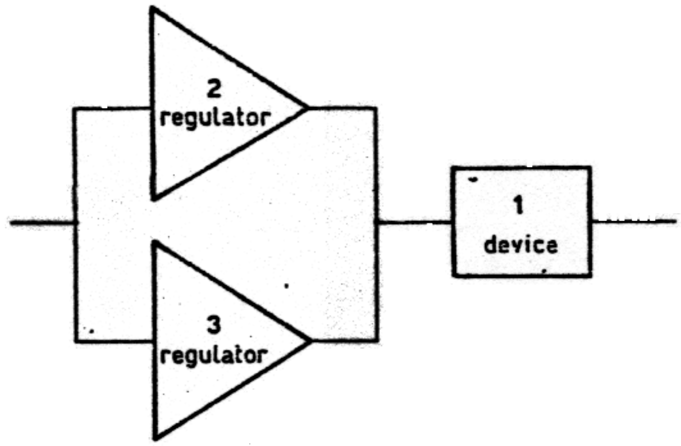

Fig. 2a. Electronic system block diagram.

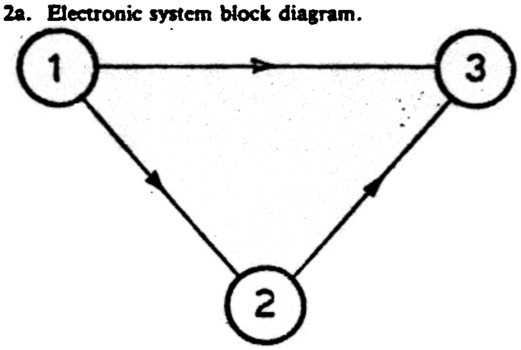

Fig. 2b. Relation graph.

$$\Lambda^{(1)} = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \times 10^{-6}. \quad \Lambda^{(2)} = \begin{matrix} -0.1 & 0 \\ 0.1 & 0 \end{matrix} \times 10^{-6}.$$

$$\Lambda^{(3)} = \begin{bmatrix} -0.05 & 0 \\ 0.05 & 0 \end{bmatrix} \times 10^{-6}.$$

$$G_{21} = G_{31} = \begin{matrix} 1 & 0 \\ 0 & 10 \end{matrix}, \; G_{32} = \begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}$$

Then:

$$\Lambda_2 = \Lambda_1 \otimes I_2 + G_{21} \otimes \Lambda^{(2)}$$

$$= \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \otimes I_2 + \begin{matrix} & 0 \\ 0 & 10 \end{matrix}$$

$$\otimes \begin{bmatrix} -0.1 & 0 \\ 0.1 & 0 \end{bmatrix} \times 10^{-6}$$

$$= \begin{bmatrix} -2-0.1 & & & \\ 0.1 & -2 & & \\ 2 & 0 & -10 \times 0.1 & \\ 0 & 2 & 10 \times 0.1 & 0 \end{bmatrix} \times 10^{-6}$$

$$\Lambda_3 = \begin{bmatrix} -2.1 & 0 & 0 & 0 \\ 0.1 & -2 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \otimes I_2 + \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} -0.05 & 0 \\ 0.05 & 0 \end{bmatrix} \times 10^{-6}/\text{hour}$$

$$\Lambda = \begin{bmatrix} -2.15 & & & & & & & \\ 0.05 & -2.1 & & & & \bigcirc & & \\ 0.1 & 0 & -2.1 & & & & & \\ 0 & 0.1 & 0.1 & -2 & & & & \\ 2 & 0 & 0 & 0 & -6 & & & \\ 0 & 2 & 0 & 0 & 5 & -1 & & \\ 0 & 0 & 2 & 0 & 1 & 0 & -10 & \\ 0 & 0 & 0 & 2 & 0 & 1 & 10 & 0 \end{bmatrix} \times 10^{-6}/\text{hour}$$

4.2 *Systems with maintenance*

The reliability or availability of a system with maintenance needs the solution of $\dot{p}(t) = \Lambda p(t)$, where Λ depends of failure and repair rates. As stated in Property 1 in section 3, Λ can be σ-decomposed into the sum of two matrices:

$$\Lambda = L + M \tag{24}$$

such that L is lower triangular and contains only failure rates and M is upper triangular and contains only repair rates. M is referred to as repair-rate matrix. A similar σ-decomposition yields:

$$\Lambda^{(i)} = L^{(i)} + M^{(i)} \tag{25}$$

for each component. The L can be regarded as the transition-rate matrix of a system without maintenance (viz. $M^{(i)} = 0$, for all i) and it can be obtained by means of the rules given in section 4.1. In general L and M are both obtained by means of Kronecker products and matrix sums; according to Property 1 and Theorem 3 this kind of operations leads to (24).

A theory of general repair-rate s-dependence is possible following the procedure used for failure rates. Nevertheless it is customary to consider only repair-rate s-dependence originating from different maintenance policies and system properties (availability or reliability) [9]. This leads to an s-dependence such that in a given state only two possibilities exist: repair or not repair. Therefore the repair rate adjustment factors are:

$$\alpha_{ij} \equiv \mu_j^{(i)}/\mu^{*(i)} \tag{26}$$

for all system states j in which component i is bad. $\mu^{*(i)}$ is the repair rate of component i in the system state in which only component i is bad.

$$\alpha_{ij} \equiv \begin{cases} 1, \text{ if a repair is allowed} \\ 0, \text{ otherwise} \end{cases} \tag{27}$$

In a system where transition failure rates are s-dependent while the repair rates are not, Λ is evaluated by means of (24), where L is defined as in (21) and M is now defined as the transition matrix for the off-line maintainability problem (14).

4.2.1 *Reliability with on-line multiple maintenance*

Repairs on bad components are allowed only if the system is good.

Theorem 6 (Reliability with on-line maintenance)

$$\text{Let } M^{(i)} \equiv \begin{bmatrix} 0 & \mu^{(i)} \\ 0 & -\mu^{(i)} \end{bmatrix} \tag{28}$$

and c be the structure vector of the system, then

$$M_R = [\textstyle\sum_{i=1}^{N} M^{(i)}] \operatorname{diag}\{c\} \tag{29}$$

where diag $\{c\}$ is a diagonal matrix whose entries are the elements of c. The multiple on-line reliability transition matrix Λ can be obtained from (24).

4.2.2 *Availability with single on-line maintenance*

This case is related to that of repairable system with only one maintenance team. Order the components in an increasing priority list: viz. only the component with higher index is repaired in a state with more than one bad component. Consider the relation graph associated with this problem. For all the couples of vertices i and r with $i < r$ there is no dependence of component r on component i. Therefore there exists no arc (n_r, n_i) for $i < r$. As a consequence the relation graph is acyclic [10]. In this case matrix M is obtained by Theorem 7.

Theorem 7 (Availability with single on-line maintenance)

$$\text{Let } W \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

Then

$$M_k = M_{k-1} \otimes W + I_{2^{k-1}} \otimes M^{(i)},\ k = i = 2, 3, \ldots, N \tag{31a}$$

$$M_1 = M^{(1)} \tag{31b}$$

where M_k is related to subsystem S_k, and $M^{(i)}$ is defined as in (28).

4.2.3 *Reliability with single on-line maintenance*

This case can be obtained by superimposing cases 4.2.1 and 4.2.2.

Theorem 8 (Reliability with single on-line maintenance)

Let M_{RS} be the repair rate matrix, then:

$$M_{RS} = M_N \cdot \text{diag}\{c\} \tag{32}$$

where M_N is obtained from recursive relation (31a).

$$M_k = M_{k-1} \otimes W + I_{2^{k-1}} \otimes M^{(i)} \quad k = i = 2, 3, \ldots, N \tag{33a}$$

$$M_1 = M^{(1)} \tag{33b}$$

and c is the structure vector.

The proof follows immediately from Theorems 5 and 6. Matrix Λ is evaluated by means of (24).

Example 3

Study the availability of the system in Figure 3

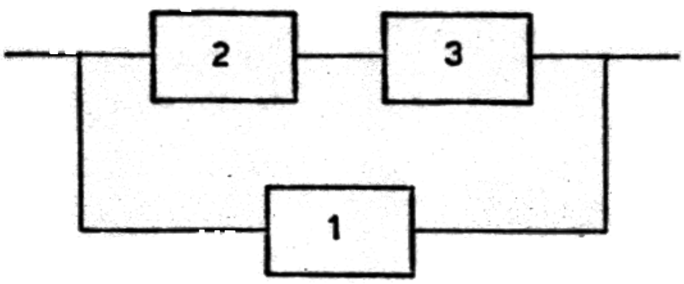

Fig. 3. Block diagram of a system with single on-line maintenance.

The components are s-independent with regard to failure. The priority list #2, #3, #1 is given; i.e. #1 is to be repaired before #3, and #3 before #2.

Let

$$L^{(i)} \equiv \begin{bmatrix} -\lambda^{(i)} & 0 \\ \lambda^{(i)} & 0 \end{bmatrix}, \quad M^{(i)} \equiv \begin{matrix} 0 & \mu^{(i)} \\ 0 & -\mu^{(i)} \end{matrix}, \text{ for } i = ', 2, 3.$$

And according to (4.1)—

$$M_2 = M_1 \otimes W + I_2 \otimes M^{(3)}$$

$$M_1 = M^{(2)}$$

$$M_2 = \begin{bmatrix} 0 & \mu^{(3)} & \mu^{(2)} & 0 \\ & -\mu^{(3)} & 0 & 0 \\ & & -\mu^{(2)} & \mu^{(3)} \\ \bigcirc & & & -\mu^{(3)} \end{bmatrix}$$

$$M_3 = M_2 \otimes W + I_2 \otimes M^{(1)}$$

$$M_3 = \begin{bmatrix} 0 & \mu^{(1)} & \mu^{(3)} & 0 & \mu^{(2)} & 0 & 0 & 0 \\ & -\mu^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & -\mu^{(3)} & \mu^{(1)} & 0 & 0 & 0 & 0 \\ & & & -\mu^{(1)} & 0 & 0 & 0 & 0 \\ & & & & -\mu^{(2)} & \mu^{(1)} & \mu^{(3)} & 0 \\ & \bigcirc & & & & -\mu^{(1)} & 0 & 0 \\ & & & & & & -\mu^{(3)} & \mu^{(1)} \\ & & & & & & & -\mu^{(1)} \end{bmatrix}$$

$$M = M_3$$

$$\Lambda = L + M,$$

The availability is obtained by integrating the equations—

$$\dot{p}(t) = \Lambda p(t), \quad A(t) = c^T p(t) \tag{34}$$

The structure vector c is obtained from the following Table I

TABLE I

$x^{(2)}$	$x^{(3)}$	$x^{(1)}$	c
	1	1	
	1	0	
	0	1	
	0	0	0
0	1	1	1
0	1	0	
0	0	0	
0	0	0	0

$$c^T = |1\,1\ \ 1\,1\,0\,1\,0|$$

Let us now compute the reliability of the same system with the same priority list.

$$L = \sum_{\oplus 1}^{3} = [L^{(2)} \oplus L^{(3)}] \oplus L^{(1)} = \begin{bmatrix} -\lambda^{(1)}-\lambda^{(2)}-\lambda^{(3)} & & & & & & & \\ \lambda^{(1)} & -\lambda^{(2)}-\lambda^{(3)} & & & & \bigcirc & & \\ \lambda^{(3)} & 0 & -\lambda^{(1)}-\lambda^{(2)} & & & & & \\ 0 & \lambda^{(3)} & \lambda^{(1)} & -\lambda^{(2)} & & & & \\ \lambda^{(2)} & 0 & 0 & 0 & -\lambda^{(1)}-\lambda^{(3)} & & & \\ 0 & \lambda^{(2)} & 0 & 0 & \lambda^{(1)} & -\lambda^{(3)} & & \\ 0 & 0 & \lambda^{(2)} & 0 & \lambda^{(3)} & 0 & -\lambda^{(1)} & \\ 0 & 0 & 0 & \lambda^{(2)} & 0 & \lambda^{(3)} & \lambda^{(1)} & 0 \end{bmatrix}$$

$$= M_3 \cdot \text{diag}\, c$$

$$M_{RS} = \begin{bmatrix} 0 & \mu^{(1)} & \mu^{(3)} & 0 & \mu^{(2)} & 0 & 0 & 0 \\ & -\mu^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & -\mu^{(3)} & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & \bigcirc & & & -\mu^{(2)} & 0 & \mu^{(3)} & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & -\mu^{(3)} & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$+ M_{RS}.$$

The reliability of the system is obtained by integrating the equations:

$$\dot{p}(t) = \Lambda p(t), \quad R(t) = c^T p(t). \tag{35}$$

5. COMPUTATIONAL REMARKS AND EXPERIMENTAL RESULTS

We are developing a large package for computer-aided evaluation of system reliability where *s*-dependence is taken into account. The results reported in this paper are used in two ways:

1. Develop a computer code to build the system transition rate matrix from component data, *s*-dependence and system structure.
2. Implement an algorithm for numerical integration of Markov equations based on decomposition techniques and using information about Λ matrix reported in the previous sections.

As to the preliminary experimental results, a few examples of different size have been tested on UNIVAC 1100/80. The following table reports for each example the number of components N, the time t_f for formulating Λ, the time t_c to build structure vector c, the time t_i for integrating $\dot{p} = \Lambda p$ (using trapezoidal rule). For all the examples the mission time has been the same (10^4 hour) and the solution has been computed each 100 hour. Table II shows that time for computing Λ and c is negligible compared with integration time. In order to reduce memory requirements a suitable data structure using sparse matrix techniques has been implemented [11]. Only non-zero values of Λ and c are stored. The use of decomposition techniques will allow reduced integration time, in some cases drastically.

APPENDIX A

Proof of Theorem 3

Let v be the index of any system state and u be the index of the system state related to v by the failure of component i. Then λ_{uv} is a failure transition rate and its value is $\lambda^{(i)}$. In order to compute v and u, the values of $x^{(i)}$ in (11) are the same for all the components $i \neq i'$ while $x^{(i)}$ is 1 for state v and 0 for state u. From (11) it follows that $u > v$. In an analogous way it can be shown that $u < v$ if λ_{uv} is a repair transition rate. *Q.E.D.*

TABLE II

Example	Number of components	t_f (sec)	t_c (sec)	t_i (sec)
1	4			
2	6			
3	8			
4	10			

Proof of Theorem 4

The zero/non-zero structure of $\Lambda = \{\lambda_{uv}\}$ depends on the transition diagram [12] and therefore not on *s*-dependence assumption. Then we can study the general case of *s*-dependence beginning from the result obtained from *s*-independent failure rates. Consider an *s*-independent system. We recall that $A^{[k]} \equiv A \otimes A \otimes \ldots \otimes A$ (Kronecker power of A) [6] and $I_2^{[k]} = I_{2^k}$. Developing (9), we get

$$\Lambda = ((\ldots(\Lambda^{(1)} \otimes I_2 + I_2 \otimes \Lambda^{(2)} \otimes I_2 + I_2 \otimes \Lambda^{(3)}$$

$$\otimes I_2 + \ldots) + I_2^{[N-1]} \otimes \Lambda^{(N)}$$

$$= \Lambda^{(1)} \otimes I_2^{[N-1]} + I_2 \otimes \Lambda^{(2)} \otimes I_2^{[N-2]}$$

$$+ \ldots + I_2^{[i-1]} \otimes \Lambda^{(i)} \otimes I_2^{[N-i]} + \ldots + I_2^{[N-1]} \otimes \Lambda^{(N)}$$

$$= \Lambda_N = \textstyle\sum_{i=1}^{N} D_N^{(i)}$$

$$D_N^{(i)} \equiv I_2^{[i-1]} \otimes \Lambda^{(i)} \otimes I_2^{[N-i]}$$

The meaning of the Kronecker product of $\Lambda^{(i)}$ with identity matrices is to insert $\lambda^{(i)}$ and $-\lambda^{(i)}$ in all the 2^{N-1} columns of matrix Λ representing the system states in which component i is good. In a *s*-dependent system, Λ matrix keeps the same structure and therefore it is still possible to express it as a sum:

$$\Lambda = \Lambda_N = \textstyle\sum_{i=1}^{N} \overline{D}_N^{(i)}$$

where matrix $\overline{D}_N^{(i)}$ is pertinent to component i and takes into account now the *s*-dependence. Suppose first that component i failure rate depends on the state of component r.

Assume $r > i$ for example.

Matrix $\overline{D}_N^{(i)}$ is obtained by a relation with the same structure as that used for $D_N^{(i)}$.

$$\overline{D}_N^{(i)} = I_2^{[i-1]} \otimes \Lambda^{(i)} \otimes I_2^{[r-i-1]} \otimes G_{ir} \otimes I_2^{[N-r]}$$

In $\overline{D}_N^{(i)}$, matrix G_{ir} takes into account the s-dependence between components i and r. This equation can be recast as:

$$\overline{D}_N^{(i)} = D_{r-1}^{(i)} \otimes G_{ir} \otimes I_2^{N-r}$$

where $D_{r-1}^{(i)} \equiv I_2^{i-1} \otimes \Lambda^{(i)}$ is the transition rate matrix of the subsystem S_{r-1} that contains only s-independent components.

Due to (20), $\overline{D}_N^{(i)}$ has among its entries the failure rate $\lambda^{(i)}$ for the states in which i and r are good and $\gamma_{ir}\lambda^{(i)}$ for the states in which i is good and r is bad.

Let us turn now to the more general case in which component i failure rate depends on more than one component. Then—

$$\overline{D}_N^{(i)} = G_{i1} \otimes G_{i2} \otimes \ldots \otimes G_{i,i-1} \otimes \Lambda^{(i)} \otimes G_{i,i+1} \otimes \ldots$$
$$= \Big(\prod_{\otimes\, r=1}^{i-1} G_{ir}\Big) \otimes \Lambda^{(i)} \otimes \Big(\prod_{\otimes\, r=i+1}^{N} G_{ir}\Big).$$

We reach the general conclusion that a failure rate $\lambda^{(i)}$ in a system state is multiplied by the stress adjustment factors γ_{ir} (for $r = 1, 2, \ldots i-1, i+1, \ldots, N$) if component r is failed or by 1 if component r is good. Equation above allows to compute $\overline{D}_N^{(i)}$ for any component i and any s-dependence. The whole matrix Λ is eventually obtained by summing $\overline{D}_N^{(i)}$ matrices. *Q.E.D*

Proof of Theorem 5

Recall that $G_{ir} = I_2$ if $i < r$; we get from (21):

$$\Lambda = \Lambda^{(1)} \otimes I_2^{N-1} + \ldots + \prod_{\otimes\, r=1}^{i-1} G_{ir} \otimes \Lambda^{(i)} \otimes I_2^{N-i}$$
$$+ \ldots + \prod_{\otimes\, r=1}^{N-1} G_{ir} \otimes \Lambda^{(N)}$$
$$= \Big[\Lambda^{(1)} \otimes I_2^{N-2} + \ldots + \prod_{\otimes\, r=1}^{i-1} G_{ir} \otimes \Lambda^{(i)} \otimes I_2^{N-i-1} + \ldots +$$
$$+ \prod_{\otimes\, r=1}^{N-2} G_{ir} \otimes \Lambda^{(N-1)} \Big] \otimes I_2 + \prod_{\otimes\, r=1}^{i-1} G_{ir} \otimes \Lambda^{(k)}$$
$$= \Lambda_{N-1} \otimes I_2 + G_{\forall S_{N-1}} \otimes \Lambda^{(N)}.$$

Q.E.D.

Proof of Theorem 6

Let

$$M \equiv \sum_{\otimes\, i=1}^{N} M^{(i)}.$$

M is the transition rate matrix for the off-line maintainability (see Theorem 2); (28) is equivalent to (14). $M \cdot \mathrm{diag}\{c\}$ is the off-line maintainability matrix where λ_{uv} are set to 0 in the failed system states; then by definition $M_N = M \cdot \mathrm{diag}\{c\}$

Proof of Theorem 7

Unilateral s-dependence leads to a recursive relation for the buildup of M_k matrix as in the case of s-dependent failure rates. In addition, since adjustment factors α_{ir} values are now 0 or 1, W is a Boolean diagonal matrix.

Consider a Boolean matrix $\overline{W}$, defined as—

$$\overline{W} = \begin{bmatrix} \overline{W}_{11} & 0 \\ 0 & \overline{W}_{22} \end{bmatrix}$$

The relation—

$$M_k = M_{k-1} \otimes \overline{W} + I_2^{k-1} \otimes M^{(i)} \quad i = k = 2, 3, \ldots, N$$

$$M_1 = M^{(1)}$$

gives the multiple-maintenance matrix in the particular case that $\overline{W} = I$. In the general case, let—

$$M'_k \equiv M_{k-1} \otimes \overline{W}.$$

It is simple matter to see that M'_k contains the repair rates of the components belonging to S_{k-1}. Two states in subsystem S_k correspond to each state of S_{k-1}. According to (11), the odd states (i.e. those represented in matrix M'_k by odd numbered columns) are good states for component i and conversely the even ones. More detail is given below in the expanded matrix for M'_k.

$$M'_k \equiv \{m'_{k_{i,j}}\} = M_{k-1} \otimes \overline{W} = \begin{bmatrix} \overline{W}_{11} m_{k-1_{1,1}} & 0 & \overline{W}_{11} m_{k-1_{1,2}} & 0 & \overline{W}_{11} m_{k-1_{1,2^{k-1}}} & 0 \\ 0 & \overline{W}_{22} m_{k-1_{1,1}} & 0 & \overline{W}_{22} m_{k-1_{1,2}} & 0 & \overline{W}_{22} m_{k-1_{1,2^{k-1}}} \\ & & \overline{W}_{11} m_{k-1_{2,2}} & 0 & \cdots & \cdots \\ & \bigcirc & 0 & \overline{W}_{22} m_{k-1_{2,2}} & & \\ & & & & \overline{W}_{11} m_{k-1_{2^{k-1},2^{k-1}}} & 0 \\ & & & & 0 & \overline{W}_{22} m_{k-1_{2^{k-1},2^{k-1}}} \end{bmatrix}$$

where $M_{k-1} \equiv \{m_{k-1_{i,r}}\}$.

With the assumption that component i is to be repaired before any other component belonging to S_{k-1}, the repair transition rates from even states are related only to component i. Therefore all repair rates for subsystem S_{k-1} are set to 0 in the even numbered states. This means that:

$$\overline{W}_{22} m'_{k-1_{i,r}} = 0, \qquad \text{for each } r \text{ odd}$$

$$\overline{W}_{11} m'_{k-1_{i,r}} = m'_{k_{2i-1,2r-1}}, \qquad \text{for each } r \text{ even}$$

and therefore

$$\overline{W} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \qquad \textit{Q.E.D.}$$

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