# ALGEBRAIC METHODS

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## Outline



- Algebraic model.
- Division and substitution.
- Kernel theory.
  - Kernel and cube extraction.
- Decomposition.

## Algebraic model



- Boolean algebra:
  - Complement.
  - Symmetric distribution laws.
  - *Don't care* sets.
- Algebraic methods:
  - Boolean functions →polynomials.
  - Expressions (*sum of product* forms).

## Algebraic division



- Given two algebraic expressions:
- $f_{quotient} = f_{dividend}/f_{divisor}$ when:
  - $f_{dividend} = f_{divisor} \cdot f_{quotient} + f_{remainder}$
  - $f_{divisor} \cdot f_{quotient} \neq 0$
  - and the support of $f_{divisor}$ and $f_{quotient}$ is disjoint.

## Example

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- Algebraic division:
  - Let $f_{dividend} = ac + ad + bc + bd + e$ and $f_{divisor} = a + b$
  - Then $f_{quotient} = c + d \quad f_{remainder} = e$
  - Because $(a + b) \cdot (c + d) + e = f_{dividend}$ and $\{a, b\} \cap \{c, d\} \ = \emptyset$.
- Non-algebraic division:
  - Let $f_i = a + bc$ and $f_j = a + b$.
  - Then $(a + b) \cdot (a + c) = f_i$ but $\{a, b\} \cap \{a, c\} \neq \emptyset$.

## An algorithm for division

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- $A = \{C_j^A \ , \ j = 1, 2, \ldots, l\}$ set of cubes (monomials) of the dividend.
- $B = \{C_i^B \ , \ i = 1, 2, \ldots, n\}$ set of cubes (monomials) of the divisor.
- Quotient $Q$ and remainder $R$ are sum of cubes (monomials).

## An algorithm for division

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$ALGEBRAIC\_DIVISION(A, B)$ {
  **for** $(i = 1$ to $n)$ {
    $D = \{C_j^A$ such that $C_j^A \supseteq C_i^B\}$;
    **if** ( $D == \emptyset$ ) **return**$(\emptyset, A)$;
    $D_i = D$ with var. in $sup(C_i^B)$ dropped ;
    **if** $i = 1$
      $Q = D_i$;
    **else**
      $Q = Q \cap D_i$;
  }
  $R = A - Q \times B$;
  **return**$(Q, R)$;
}

## Example

$f_{dividend} = ac + ad + bc + bd + e;$
$f_{divisor} = a + b;$

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- $A = \{ac, ad, bc, bd, e\}$and $B = \{a, b\}$.
- $i = 1$:
  - $C_1^B = a, \ D = \{ac, ad\}$ and $D_1 = \{c, d\}$.
  - Then $Q = \{c, d\}$.
- $i = 2 = n$:
  - $C_2^B = b, \ D = \{bc, bd\}$ and $D_2 = \{c, d\}$.
  - Then $Q = \{c, d\} \cap \{c, d\} = \{c, d\}$.
- Result:
  - $Q = \{c, d\}$ and $R = \{e\}$. $f_{quotient} = c + d$ and $f_{remainder} = e$.

## Theorem

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- Given $f_i$ and $f_j$, then $f_i/f_j$ is empty when:
  - $f_j$ contains a variable not in $f_i$.
  - $f_j$ contains a cube whose support is not contained in that of any cube of $f_i$.
  - $f_j$ contains more terms than $f_i$.
  - The count of any variable in $f_j$ than in $f_i$.

## Substitution

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- Consider expression pairs.
- Apply division (in any order).
- If quotient is not void:
  - Evaluate area/delay gain
  - Substitute $f_{dividend}$ by $j \cdot f_{quotient} + f_{remainder}$ where $j = f_{divisor}$.
- Use filters to reduce divisions.

## Substitution algorithm

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```
SUBSTITUTE( G_n(V,E) ){
  for (i = 1,2,...,|V|) {
      for (j = 1,2,...,|V|; j ≠ i) {
          A = set of cubes of f_i;
          B = set of cubes of f_j;
          if (A,B pass the filter test ) {
              (Q,R) = ALGEBRAIC_DIVISION(A,B)
              if (Q ≠ ∅) {
                  f_quotient = sum of cubes of Q;
                  f_remainder = sum of cubes of R;
                  if ( substitution is favorable)
                      f_i = j · f_quotient + f_remainder;
              }
          }
      }
  }
}
```

## Extraction

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- Search for common sub-expressions:
  - Single-cube extraction: monomial.
  - Multiple-cube (kernel) extraction.
- Search for appropriate divisors.

## Definitions



- *Cube-free* expression:
  - Cannot be factored by a cube.
- *Kernel* of an expression:
  - Cube-free quotient of the expression divided by a cube, called *co-kernel*.
- *Kernel set* $K(f)$ of an expression:
  - Set of kernels.

## Example
$$f_x = ace + bce + de + g$$



- Divide $f_x$ by $a$. Get $ce$. Not cube free.
- Divide $f_x$ by $b$. Get $ce$. Not cube free.
- Divide $f_x$ by $c$. Get $ae + be$. Not cube free.
- Divide $f_x$ by $ce$. Get $a + b$. Cube free. **Kernel!**
- Divide $f_x$ by $d$. Get $e$. Not cube free.
- Divide $f_x$ by $e$. Get $ac+bc+d$. Cube free. **Kernel!**
- Divide $f_x$ by $g$. Get $1$. Not cube free.
- Expression $f_x$ is a kernel of itself because cube free.
- $K(f_x) = \{(a+b); (ac+bc+d); (ace+bce+de+g)\}$.

## Theorem
## (Brayton and McMullen)



- Two expressions $f_a$ and $f_b$ have a common multiple-cube divisor $f_d$ if and only if:
  - there exist kernels $k_a \in K(f_a)$ and $k_b \in K(f_b)$ s.t. $f_d$ is the sum of 2 (or more) cubes in $k_a \cap k_b$.
- Consequence:
  - If kernel intersection is void, then the search for common sub-expression can be dropped.

## Example



$$\begin{aligned} f_x &= ace + bce + de + g \\ f_y &= ad + bd + cde + ge \\ f_z &= abc \end{aligned}$$

- $K(f_x) = \{(a+b); (ac+bc+d); (ace+bce+de+g)\}$.
- $K(f_y) = \{(a+b+ce); (cd+g); (ad+bd+cde+ge)\}$.
- The kernel set of $f_z$ is empty.
- Select intersection $(a+b)$

$$\begin{aligned} f_w &= a + b \\ f_x &= wce + de + g \\ f_y &= wd + cde + ge \\ f_z &= abc \end{aligned}$$

## Kernel set computation



- Naive method:
  - Divide function by elements in power set of its support set.
  - Weed out non cube-free quotients.
- Smart way:
  - Use recursion:
    - Kernels of kernels are kernels.
  - Exploit commutativity of multiplication.

## Recursive kernel computation simple algorithm



```
R_KERNELS(f){
    K = ∅;
    foreach variable x ∈ sup(f) {
        if(|CUBES(f, x)| ≥ 2) {
            f^C = largest cube containing x,
              s.t. CUBES(f, C) = CUBES(f, x);
            K = K ∪ R_KERNELS(f/f^C);
        }
    }
    K = K ∪ f;
    return(K);
}

CUBES(f, C){
    return the cubes of f whose support ⊇ C;
}
```

## Analysis



- Some computation may be redundant:
  - Example:
    - Divide by $a$ and then by $b$.
    - Divide by $b$ and then by $a$.
  - Obtain duplicate kernels.
- Improvement:
  - Keep a *pointer* to literals used so far.

## Recursive kernel computation



```
KERNELS(f, j){
    K = ∅;
    for i = j to n {
        if(|CUBES(f, x_i)| ≥ 2) {
            f^C = largest cube containing x,
              s.t. CUBES(f, C) = CUBES(f, x_i);
            if (x_k ∉ C ∀k < i)
                K = K ∪ KERNELS(f/f^C, i + 1);
        }
    }
    K = K ∪ f;
    return(K);
}
```

## Example
$f = ace + bce + de + g$



- Literals $a$ or $b$. No action required.
- Literal $c$. Select cube $ce$:
  - Recursive call with arguments: $(ace+bce)/ce = a + b$; pointer $j = 3 + 1$.
  - Call considers variables $\{d, e, g\}$. No kernel.
  - Adds $a + b$ to the kernel set at the last step.
- Literal $d$. No action required.
- Literal $e$. Select cube $e$:
  - Recursive call with arguments: $ac + bc + d$ and pointer $j = 5 + 1$.
  - Call considers variable $\{g\}$. No kernel.
  - Adds $ac + bc + d$ to the kernel set at the last step.
- Literal $g$. No action required.
- Adds $ace + bce + de + g$ to the kernel set.
- $K = \{(ace + bce + de + g), (ac + bc + d), (a + b)\}$.

## Matrix representation of kernels



- Boolean matrix:
  - Rows: cubes. Columns: variables.
- Rectangle $(R, C)$:
  - Subset of rows and columns with all entries equal to 1.
- Prime rectangle:
  - Rectangle not inside any other rectangle.
- Co-rectangle $(R, C')$ of a rectangle $(R, C)$:
  - $C'$ are the columns not in $C$.
- A co-kernel corresponds to a prime rectangle with at least two rows.

## Example
$f_x = ace + bce + de + g$

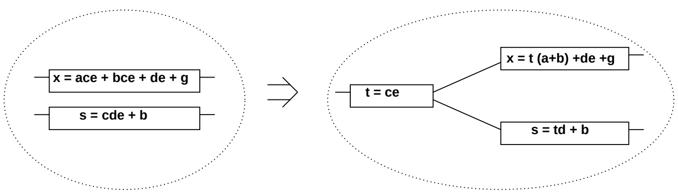


| | var | $a$ | $b$ | $c$ | $d$ | $e$ | $g$ |
|---|---|---|---|---|---|---|---|
| cube | $R \backslash C$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $ace$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $bce$ | 2 | 0 | 1 | 1 | 0 | 1 | 0 |
| $de$ | 3 | 0 | 0 | 0 | 1 | 1 | 0 |
| $g$ | 4 | 0 | 0 | 0 | 0 | 0 | 1 |

- Rectangle (prime): $(\{1, 2\}, \{3, 5\})$
  - Co-kernel $ce$.
- Co-rectangle: $(\{1, 2\}, \{1, 2, 4, 6\})$.
  - Kernel $a + b$.

## Single-cube extraction





## Single-cube extraction



- Form *auxiliary* function:
  - Sum of all local functions.
- Form matrix representation:
  - A rectangle with two rows represents a *common cube.*
  - Best choice is a prime rectangle.
- Use function ID for cubes:
  - Cube intersection from different functions.

## Example



- Expressions:
  - $f_x = ace + bce + de + g$
  - $f_s = cde + b$
- Auxiliary function:
  - $f_{aux} = ace + bce + de + g + cde + b$
- Matrix:

| | | var | $a$ | $b$ | $c$ | $d$ | $e$ | $g$ |
|---|---|---|---|---|---|---|---|---|
| cube | ID | $R \backslash C$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $ace$ | x | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $bce$ | x | 2 | 0 | 1 | 1 | 0 | 1 | 0 |
| $de$ | x | 3 | 0 | 0 | 0 | 1 | 1 | 0 |
| $g$ | x | 4 | 0 | 0 | 0 | 0 | 0 | 1 |
| $cde$ | s | 5 | 0 | 0 | 1 | 1 | 1 | 0 |
| $b$ | s | 6 | 0 | 1 | 0 | 0 | 0 | 0 |

- Prime rectangle: $(\{1, 2, 5\}, \{3, 5\})$
- Extract cube $ce$.

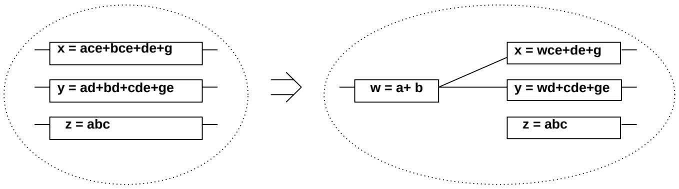
## Cube extraction algorithm



$CUBE\_EXTRACT(\ G_n(V, E)\ )\{$
  **while** (some favorable common cube exist) {
    $C$ = select common cube to extract;
    Generate new label $l$;
    Add to network $v_l$ and $f_l = f^C$;
    Replace all functions $f$, where $f_l$ is a divisor,
     by $l \cdot f_{quotient} + f_{remainder}$;
    }
  }

## Multiple-cube extraction





## Multiple-cube extraction



- We need a kernel/cube matrix.
- Relabeling:
  - Cubes by new variables.
  - Kernels by cubes.
- Form *auxiliary* function:
  - Sum of all kernels.
- Extend cube intersection algorithm.

## Example



- $f_p = ace + bce$.
  - $K(f_p) = \{(a + b)\}$.
- $f_q = ae + be + d$.
  - $K(f_q) = \{(a + b); (ae + be + d)\}$.
- Relabeling:
  - $x_a = a; x_b = b; x_{ae} = ae; x_{be} = be; x_d = d;$
    - $K(f_p) = \{\{x_a, x_b\}\}$
    - $K(f_q) = \{\{x_a, x_b\}; \{x_{ae}, x_{be}, x_d\}\}$.

## Example (2)



- $f_{aux} = x_a x_b + x_a x_b + x_{ae} x_{be} x_d$.
- Co-kernel: $x_a x_b$.
  - $x_a x_b$ corresponds to kernel intersection $a + b$.
  - Extract $a + b$ from $f_p$ and $f_q$.

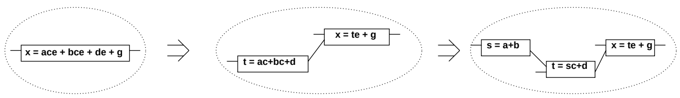
## Kernel extraction algorithm



$KERNEL\_EXTRACT(\ G_n(V,E)\ ,\ n, k)\{$
  **while** (some favorable common kernel intersection exist)
    Compute kernel set of level $\leq k$;
    **for** $(i = 1$ to $n)$ $\{$
      Compute kernel intersections;
      $f$ = select kernel intersection to extract;
      Generate new label $l$;
      Add $v_l$ to the network with expression $f_l = f$;
      Replace all functions $f$ where $f_l$ is a divisor
        by $l \cdot f_{quotient} + f_{remainder}$;
      $\}$
    $\}$
  $\}$

## Decomposition






## Decomposition



- Different ways:
  - Method of Ashenhurst and Curtis.
  - NAND/NOR decomposition.
- Kernel-based decomposition:
  - Divide expression recursively.

## Example
$f_x = ace + bce + de + g$



- Select kernel $ac + bc + d$.
- Decompose: $f_x = te + g$; $f_t = ac + bc + d$;
- Recur on the quotient $f_t$:
  - Select kernel $a + b$:
  - Decompose: $f_t = sc + d$; $f_s = a + b$;

## Decomposition algorithm



$DECOMPOSE($ $G_n(V, E)$ , $k)\{$
    **repeat** $\{$
        $v_x =$ selected vertex with expression
          whose size is above $k$;
        **if** $(v_x = \emptyset)$ **return**;
        decompose expression $f_x$;
    $\}$
$\}$

## Summary
## Algebraic transformations



- View Boolean functions as algebraic expression.
- Fast manipulation algorithms.
- Some optimality lost,
  because Boolean properties are neglected.
- Useful to reduce large networks.